Entrance Examination
Department of Fusion Science
School of Physical Sciences
Graduate University for Advanced Studies

Subject No.	Mathematics (1)

Answer the following questions. Note that n is a positive integer, $(2n)!! = 2 \times 4 \times 6 \times \cdots \times 2n$, and $(2n-1)!! = 1 \times 3 \times 5 \times \cdots \times (2n-1)$.

(1) Obtain the relation between I_n and I_{n+2} for the indefinite integral: $I_n = \int \sin^n x dx$.

(2) Using the result from question (1), evaluate the definite integral: $K_n = \int_0^{\pi/2} \sin^n x dx$.

(3) The Beta function and the Gamma function are defined by

$$\begin{aligned} \mathrm{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x>0, y>0), \\ \text{and} \quad \Gamma(x) &= \int_0^\infty e^{-t}t^{x-1}dt \quad (x>0), \end{aligned}$$

respectively. They are related such that:

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Prove the following relations:

(a) $\mathrm{B}(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta d\theta$,

(b) $\Gamma(n+1) = n!$,

(c) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,

(d) $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$.

(4) Using the results from question (3), evaluate the definite integral: $H_n = \int_0^{\pi/2} \cos^n x dx$.

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Subject No.	Mathematics (2)

Solve the following differential equations by taking steps described below.

$$\begin{cases} \dfrac{dx}{dt} &= x + 2y \\ \dfrac{dy}{dt} &= -2x + y \end{cases}.$$

(1) If these equations are written in the following manner:

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

calculate the eigenvalues λ_1 and λ_2 of matrix A and the eigenvectors $\boldsymbol{u}_1 = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\boldsymbol{u}_2 = \begin{pmatrix} r \\ s \end{pmatrix}$ of matrix A.

(2) Provided that $B = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, $X(t)$ and $Y(t)$ are defined as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Derive the appropriate expressions for dX/dt and dY/dt.

(3) Obtain the general solution of $x(t)$ and $y(t)$ using the result from question (2).

(4) Obtain $x(t)$ and $y(t)$, assuming that $x(0) = 4$ and $y(0) = 2$.

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Subject No.	Physics (1)

Consider a pendulum in small oscillation on the Earth. The pendulum is made of a bob of mass m suspended from a fixed point with a massless string. Here, the air resistance and sliding friction at the pivot point may be ignored. When observed from the frame of reference rotating with the Earth at a constant angular velocity Ω, the equation of motion of the bob is given by:

$$m\frac{d^2\boldsymbol{R}}{dt^2} = \boldsymbol{f} - m\boldsymbol{g} - 2m\boldsymbol{\Omega} \times \frac{d\boldsymbol{R}}{dt}, \qquad \text{(A)}$$

where $\boldsymbol{R}$ represents the position of the bob, $\boldsymbol{f}$ is the restoring force, $m\boldsymbol{g}$ is the gravity, and $\boldsymbol{\Omega}$ is the angular velocity vector, the magnitude of which is Ω. Note that $m\boldsymbol{g}$ is the sum of the attractive force from the Earth and the centrifugal force due to the Earth's rotation. Here, $m\boldsymbol{g}$ is regarded as a constant. The Cartesian coordinate system (x, y, z) fixed on the Earth, the origin O, and the latitude θ at O are defined as shown in the Fig. 1. The pivot point of the pendulum is set at a certain point on the positive z axis. $\boldsymbol{R}$ in Eq.(A) is expressed by $\boldsymbol{R} = (x, y, z)$. Answer the following questions.

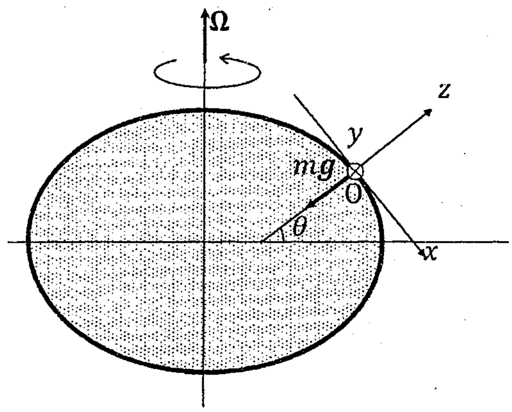

Figure 1: The bold solid line shows the surface of the Earth. The latitude θ at O is defined as shown in the figure.

(1) $z = 0$ and $dz/dt = 0$ are assumed because of small oscillation. The restoring force is proportional to the displacement, so that $\boldsymbol{f} = (-kx, -ky, 0)$, where k is a constant. Derive the equations of motion of the bob in the x and y directions.

(2) Prove that $d\boldsymbol{L}/dt = \boldsymbol{R} \times \boldsymbol{F}$, where $\boldsymbol{L} = \boldsymbol{R} \times (md\boldsymbol{R}/dt)$ is the angular momentum of the bob and $\boldsymbol{F}$ is the total force acting on the bob. And also express dL_z/dt in terms of x, y, dx/dt, dy/dt, m, Ω and θ using the results from question (1). Here L_z denotes the z component of $\boldsymbol{L}$.

(3) Express L_z in terms of x, y, m, Ω and θ integrating dL_z/dt obtained in question (2) under the assumption that the bob passes the origin at $t = 0$.

(4) Prove $L_z = mr^2 d\phi/dt$ from the definition of $\boldsymbol{L}$ using the polar coordinates (r, ϕ) defined by $(x, y) = (r\cos\phi, r\sin\phi)$. And also express the angular speed $d\phi/dt$ in terms of Ω and θ using the expression for L_z obtained in question (3).

(5) Express r in terms of t, k, m, Ω, θ and v_0 solving the equation of motion for r under the assumption that $r = 0$ and $dr/dt = v_0$ at $t = 0$.

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Subject No.	Physics (2)

Consider a system of N non-interacting and distinguishable particles, where N is an even integer. They are fixed in position, but can individually take an energy level of either 0 or ε, where ε is a positive value. Under these conditions, the total energy of this system is thus given by the relation: $E = n\varepsilon$, where E is the total energy of this system and n is the number of particles at the energy level of ε. Then, obtain the relation between the specific heat, C, and the absolute temperature, T, of this system, taking the following steps:

(1) Assuming that the entropy of this system is given by the relation: $S = k \ln \Omega(E)$, where S is the entropy of this system and k is the Boltzmann constant, what would $\Omega(E)$ mean?

(2) Express S in terms of N and n.

(3) Derive Stirling's approximation formula: $\ln M! \approx M \ln M - M$, where M is a large positive integer ($M \gg 1$).

Related to the above-mentioned system, answer the following questions assuming that $N \gg 1$ and thus E can be regarded as a continuous variable.

(4) Rewrite S using Stirling's approximation derived in question (3), and also find the value of n for which S is maximized.

(5) Define the range of n for which T can take positive values.

(6) Obtain the relation between C and T under the condition(s) defined by question (5).

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Subject No.	Physics (3)

Consider a one-dimensional quantum-mechanical system described by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)$$

with the Hamiltonian:

$$H = \frac{1}{2m}p^2 + V(x),$$

where $\hbar = h/2\pi$ is Planck's constant h divided by 2π, $\psi(x,t)$ is the wave function of position x and time t, $p = -i\hbar\partial/\partial x$ is the momentum, m is the mass, and $V(x)$ represents the potential energy. The mean value of an arbitrary dynamical variable, $A(x,p)$, which does not depend on t explicitly, is given by

$$\langle A\rangle = \int_{-\infty}^{\infty}\psi^*(x,t)A(x,p)\psi(x,t)dx,$$

where $\psi^*(x,t)$ denotes the complex conjugate of $\psi(x,t)$ and $\int_{-\infty}^{\infty}\psi^*(x,t)\psi(x,t)dx = 1$ is assumed. Note that $\langle A\rangle$ is a function of t.

(i) Show that $\langle A\rangle$ satisfies

$$i\hbar\frac{d}{dt}\langle A\rangle = \langle[A,H]\rangle.$$

Here, the commutator of A and H is defined by $[A,H] = AH - HA$.

(ii) Consider a one-dimensional quantum-mechanical harmonic oscillator for which the potential energy is given by $V(x) = m\omega^2x^2/2$. Derive the ordinary differential equations to describe the time dependence of the mean values $\langle x\rangle$ and $\langle p\rangle$.

(iii) For the harmonic oscillator considered in question (ii), X, Y, and Z are defined by

$$X = \langle x^2\rangle - \langle x\rangle^2, \qquad Y = \langle p^2\rangle - \langle p\rangle^2, \qquad \text{and} \qquad Z = \langle xp + px\rangle - 2\langle x\rangle\langle p\rangle,$$

respectively. Under these conditions, prove that the following relations hold:

$$\frac{dX}{dt} = \frac{Z}{m}, \qquad \frac{dY}{dt} = -m\omega^2 Z, \qquad \text{and} \qquad \frac{dZ}{dt} = 2\left(-m\omega^2X + \frac{Y}{m}\right).$$

(iv) Obtain the general solution of the ordinary differential equations for $X(t)$, $Y(t)$, and $Z(t)$ shown in question (iii). Show that the conditions to maintain $X(t)$ and $Y(t)$ constant are:

$$m^2\omega^2X(0) = Y(0) \qquad \text{and} \qquad Z(0) = 0.$$

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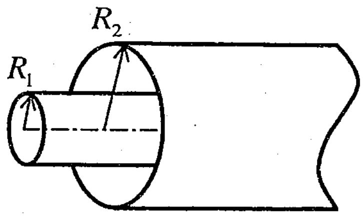
Subject No.	Engineering (1)

Consider a co-axial cylindrical capacitor of infinite length with inner and outer conductors with their respective radii of R_1 and R_2. The gap between these conductors is filled with air. Answer the following questions, assuming that the dielectric constant of air is approximated by the one in vacuum ($=\varepsilon_0$).



(1) Express the local electric field in terms of radius r, incorporating the charge per unit length q.

(2) Define the condition where the local electric field between the two conductors does not exceed the breakdown field of the air E_b, using q, R_1 and R_2. Also express by E_b the maximum allowable voltage V_b between the conductors.

(3) When the radius of the outer conductor R_2 is fixed, determine the radius of the inner conductor R_1 to maximize the allowable voltage between the conductors. Also evaluate the maximum allowable voltage.

(4) Express the stored energy per unit length in the capacitor in terms of q, R_1 and R_2.

(5) When the radius of the outer conductor R_2 is fixed, determine the radius of the inner conductor to maximize the stored energy before the breakdown. Also evaluate the maximum stored energy per unit length.

(6) For the case that $E_b = 1.0$ MV/m and $R_2 = 1.0$ cm, compute the maximum allowable voltage for (3) and the maximum stored energy per unit length for (5) to two digits accuracy.

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Subject No.	Engineering (2)

An inductor of resistance R and self-inductance L is connected in series with a non-inductive resistor of resistance R_0 to a constant potential difference V as shown in Fig. 1. The switches S_1 and S_2 are initially open.

(1) Find the expression for the potential difference V_{pq} across the inductor as a function of time t after the switch S_1 is closed.

(2) After the current in the circuit has reached its final steady-state value, the switch S_2 is closed. Find the expression for the current in S_2 as a function of time t after S_2 is closed.

(3) The inductor consists of a toroidal solenoid having a major radius $r = 300$ mm and minor radius $a = 50$ mm (see Fig. 2). A copper wire of resistivity 2×10^{-8} Ωm and diameter 0.4 mm is wound 10000 turns on a non-magnetic bobbin. Find the self-inductance L of this inductor using a solution for $r \gg a$. Also find the resistance R of the inductor. Approximate π as 3.

(4) Using the self-inductance and resistance of the inductor obtained above, and assuming that $V = 100$ V and $R_0 = 500$ Ω, what will be the magnitude and direction of the current in the switch S_2 at 0.001 second after S_2 is closed? At this point in time, the switch S_1 is opened. What will be the magnitude and direction of the current in S_2 at 0.001 second after S_1 is opened?

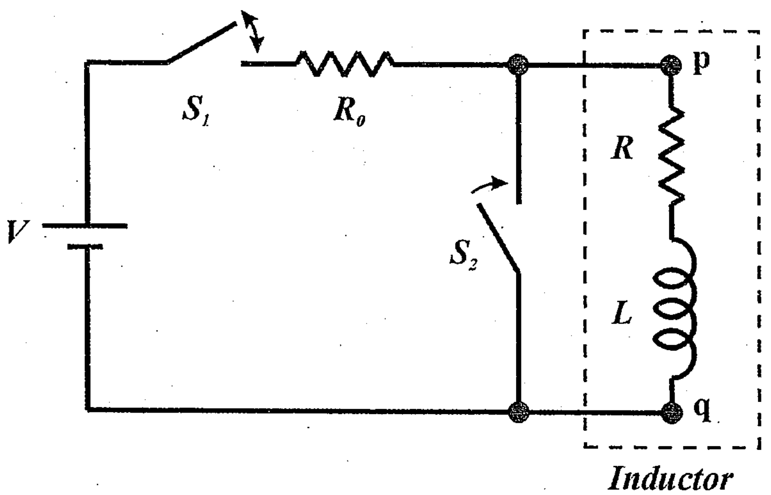

Fig. 1

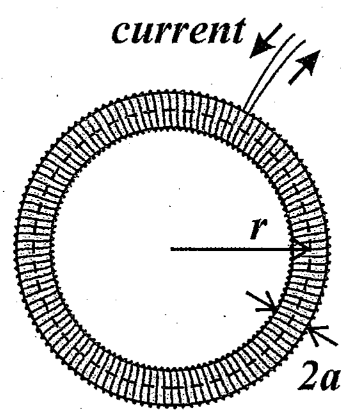

Fig. 2

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Subject No.	Engineering (3)

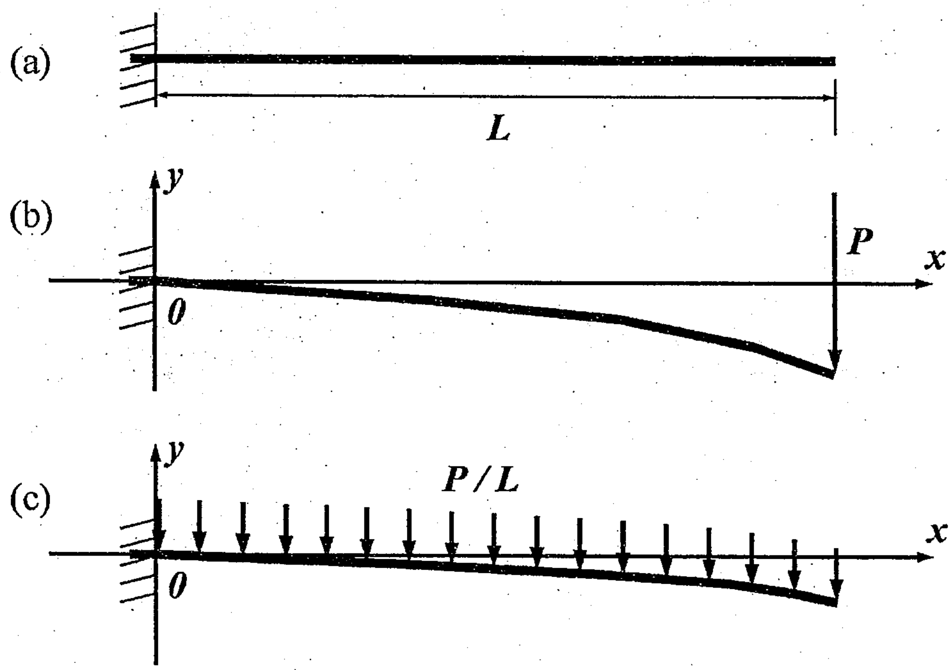
Consider a horizontal cantilever with a beam of length, L, as shown in Figure (a). The cantilever is loaded at its end by a point force, P, as shown in Figure (b).
Set the original point of x and y coordinates at the left end of the beam. Assume that Young's modulus of the beam, E, and the moment of inertia of cross sectional area of the beam, I, are constant. Neglect the weight of the beam.

(1) Express the bending moment for the cantilever by position x.

(2) Determine the maximum deflection and the maximum angle of deflection of the cantilever.

Next, consider a case in which the cantilever is subjected to a uniformly distributed load, P/L, as shown in Figure (c).

(3) Determine the maximum deflection and the maximum angle of deflection of the cantilever.



Figure