Entrance Examination
Department of Fusion Science
School of Physical Sciences
Graduate University for Advanced Studies

Subject No.	Mathematics (1)

Consider the matrix **A** and the vector **b**,

$$\mathbf{A}=\begin{pmatrix}-1 & 1 & 0 & -1/2\\ -3 & 2 & -1 & 1/2\\ 2 & 0 & 1 & 2\\ -1 & -3/2 & 0 & -3\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}2\\ 4\\ 2\\ -3\end{pmatrix}.$$

(1) Find the lower- (**L**) and upper- (**U**) triangular matrices that satisfy $\mathbf{A}=\mathbf{LU}$, where

$$\mathbf{L}=\begin{pmatrix}l_{11} & 0 & 0 & 0\\ l_{21} & l_{22} & 0 & 0\\ l_{31} & l_{32} & l_{33} & 0\\ l_{41} & l_{42} & l_{43} & l_{44}\end{pmatrix},\quad \mathbf{U}=\begin{pmatrix}1 & u_{12} & u_{13} & u_{14}\\ 0 & 1 & u_{23} & u_{24}\\ 0 & 0 & 1 & u_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

(2) Find the solution, **x**, to the simultaneous linear equations $\mathbf{Ax}=\mathbf{b}$.

Subject No.	Mathematics (2)

Consider the general solution for the differential equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 2\alpha\frac{\mathrm{d}u}{\mathrm{d}t} + \omega^2 u = A\cos(\Omega t), \qquad (*)$$

where α, ω, A, and Ω are real constants, $\omega > \alpha > 0$, u is a real function, and t is a real variable. The general solution is given by $u = u_h + u_p$, where u_h is the general solution to the homogeneous equation and u_p is the particular solution to the inhomogeneous equation $(*)$.

(1) Derive the general solution to the homogeneous equation, u_h, letting the right-hand side of equation $(*)$ be zero.

(2) Derive the particular solution, u_p, making the ansatz: $u_p = B\cos(\Omega t) + C\sin(\Omega t)$, where B and C are constants.

(3) Under the condition that $t \gg 1/\alpha$, derive the following relation:

$$\left\langle 2\alpha\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2\right\rangle = \left\langle A\frac{\mathrm{d}u}{\mathrm{d}t}\cos(\Omega t)\right\rangle,$$

where the average of a quantity X, $\langle X\rangle$, is taken over the period $T = 2\pi/\Omega$.

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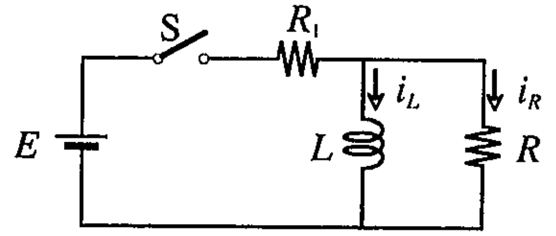
Subject No.	Science and Engineering (1)

1. In the network shown in Fig. 1, the switch S is closed at $t=0$. Calculate the currents i_R and i_L flowing in the resistance R and the inductance L, respectively, as functions of t. E is the electric potential and R_1 is the different resistance from R.

Fig. 1

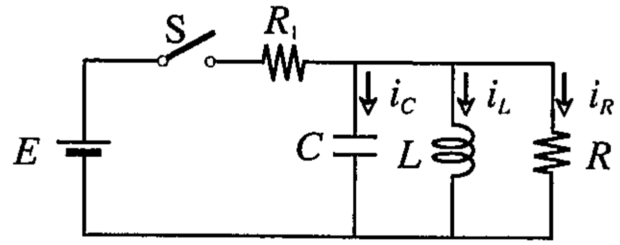


2. After S is closed at $t=0$, S is opened again at $t=t_1$. Calculate i_R and i_L as functions of t.

3. Show graphs of i_R and i_L for the entire sequence after $t=0$ through $t=t_1$.

4. In the network shown in Fig. 2, the switch S is closed at $t=0$. Derive the relations between currents i_R, i_L and i_C flowing in the resistance R, the inductance L and the capacitor C, respectively. Assume the charge in the capacitor C is zero at $t=0$.

Fig. 2



5. Show the condition to give damped-oscillations of the currents i_R, i_L and i_C using values of the circuit elements.

6. Calculate the currents i_R, i_L and i_C as functions of t in this condition.

Subject No.	Science and Engineering (2)

1) A 1 mole ideal gas system undergoes an isothermal process with temperature, T. What is the work W expended when the volume is changed from V_1 to V_2?

2) An ideal gas system undergoes an adiabatic process. Derive the equations $TV^{\gamma-1} =$ constant and $PV^{\gamma} =$ constant, where T, V and P are the temperature, the volume and the pressure, and γ is the heat capacity ratio. The relationship between the molar heat capacity at constant volume, C_v, the molar heat capacity at constant pressure, C_p, and the gas constant, R, is given as $C_p - C_v = R$.

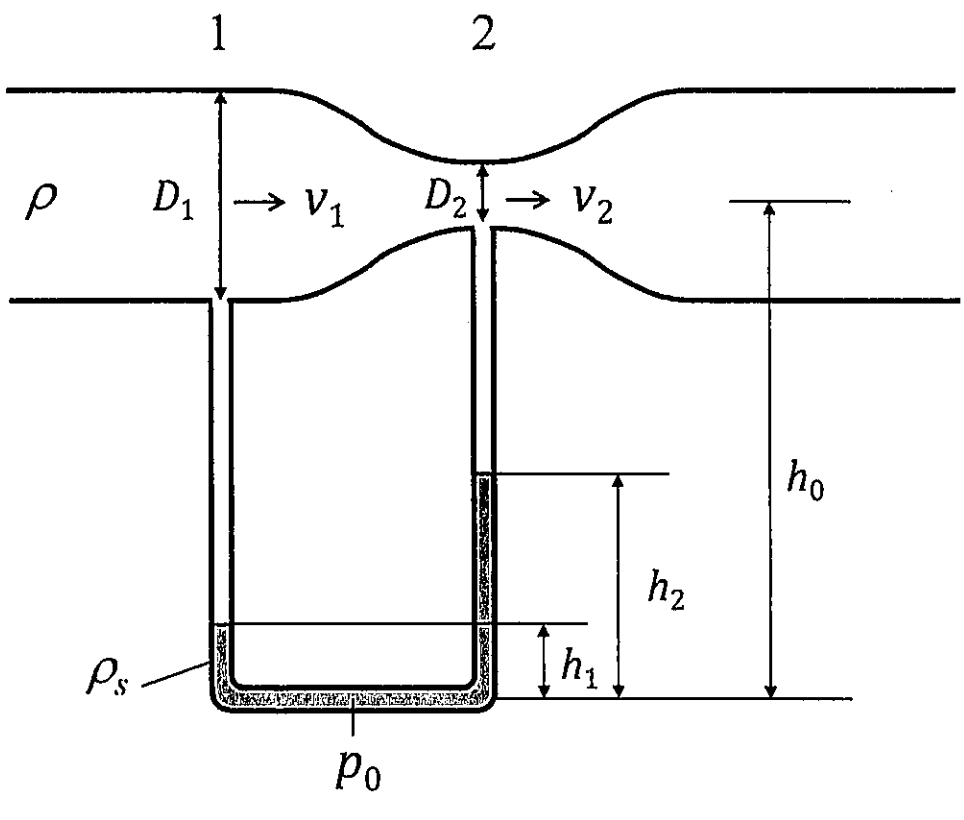
3) Consider the Carnot cycle for an ideal gas which has a high temperature heat source at temperature, T_1, and a low temperature heat source at temperature, T_2. Derive the heat efficiency which is expressed as $\eta = 1 - T_2/T_1$.

4) The Van der Waals equation for a gas of 1 mole is written as $(P + a/V^2)(V - b) = RT$. Derive the internal energy as a function of the temperature, T, and the volume, V, where a and b are constants and R is the gas constant. Assume that the molar heat capacity at constant volume, C_v, is constant.

Subject No.	Science and Engineering (3)

A fluid with mass density ρ is flowing into a Venturi tube with a manometer, and the Venturi tube is placed horizontally as shown in the figure. The diameters of the tube, velocities and pressures at the cross-sections 1 and 2 are D_1, D_2, v_1, v_2 and p_1, p_2, respectively. The manometer tube contains a fluid with the mass density ρ_s, and the heights at the cross-sections 1 and 2 are h_1 and h_2, respectively. The height of the central axis of the Venturi tube is h_0. Answer the following questions. Here g is gravitational acceleration. The energy loss, viscosity and compressibility of the flow are negligible.

1) Write the continuity equation and Bernoulli's equation at the cross-sections 1 and 2 in the Venturi tube. Write the expression for the fluid velocity, v_1, by using D_1, D_2, p_1, p_2 and ρ.
2) Write the expressions for the pressure at the bottom of the manometer, p_0, and the difference in the pressures, $p_1 - p_2$.
3) Write the expression for the volume flow, Q, without using p_1 and p_2.
4) Assume that water flows at 600 liters per minute into the Venturi tube, while the manometer contains mercury with a specific gravity of 13.6. What is the difference in the heights of the mercurial columns $h_2 - h_1$ when $D_1 = 0.21$ m and $D_2 = 0.11$ m? Assume $g = 9.8$ m^2/s.



Subject No.	Science and Engineering (4)

Consider a free particle of mass m that moves in one dimension. Its initial ($t = 0$) wave function is

$$\psi(x,0) = \left(\frac{a}{\pi}\right)^{1/4} \exp\left(i\frac{p_0}{\hbar}x - \frac{ax^2}{2}\right),$$

where $i = \sqrt{-1}$, x is the particle position, and $a(>0)$ and p_0 are real parameters. $\hbar = h/(2\pi)$ is Planck's constant, h, divided by 2π. The wave function at time t is given by

$$\psi(x,t) = \psi(x,0)\exp\left(-i\frac{H}{\hbar}t\right),$$

where H is the Hamiltonian of the particle. The Hamiltonian is given as $H = p^2/(2m)$, where p is the momentum. The expectation value of an arbitrary dynamical variable $A(x,p)$ at time t is given by

$$\langle A\rangle_t = \int_{-\infty}^{\infty} \psi^*(x,t)A(x,p)\psi(x,t)dx,$$

where $\psi^*(x,t)$ denotes the complex conjugate of $\psi(x,t)$. We can use the Gaussian integral

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2 - iqx)dx = \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left(-\frac{q^2}{4\alpha}\right),$$

where the real part of a complex number α is $\mathrm{Re}(\alpha) > 0$ and q is a real number. Complete the following:

(1) Calculate the expectation values, $\langle x\rangle_0$ and $\langle x^2\rangle_0$, at $t = 0$.

(2) Calculate the momentum wave function, $\tilde{\psi}(p,0)$, at $t = 0$. Normalize the obtained $\tilde{\psi}(p,0)$.

(3) Calculate the expectation values, $\langle p\rangle_0$ and $\langle p^2\rangle_0$, at $t = 0$.

(4) For the uncertainties of particle position and momentum at $t = 0$, we have $(\Delta x)_0^2 \equiv \langle x^2\rangle_0 - \langle x\rangle_0^2$ and $(\Delta p)_0^2 \equiv \langle p^2\rangle_0 - \langle p\rangle_0^2$, respectively. Calculate $(\Delta x)_0^2(\Delta p)_0^2$.

(5) Calculate the momentum wave function, $\tilde{\psi}(p,t)$, at all times $t > 0$. Normalize the obtained $\tilde{\psi}(p,t)$.

(6) Using the result from question (5), the wave function, $\psi(x,t)$, at all times $t > 0$ is obtained as

$$\psi(x,t) = \left(\frac{a}{\pi}\right)^{1/4} \frac{1}{\sqrt{z}} \exp\left(i\frac{p_0}{\hbar z}x - \frac{ax^2}{2z} - \frac{ip_0^2 t}{2m\hbar z}\right),$$

where $z = 1 + i\hbar at/m$ and $|z|^2 = 1 + (\hbar at/m)^2$. For the uncertainties of particle position and momentum at all times $t > 0$, we have $(\Delta x)_t^2 \equiv \langle x^2\rangle_t - \langle x\rangle_t^2$ and $(\Delta p)_t^2 \equiv \langle p^2\rangle_t - \langle p\rangle_t^2$, respectively. Verify the validity of the uncertainty relation, $(\Delta x)_t^2(\Delta p)_t^2 \geq \hbar^2/4$.